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Technical Notes

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Master Dynamic Stability Formula for Structural Members Subjected to Periodic Loads

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Nomenclature

$[G]$	=	system geometric stiffness matrix
$[K]$	=	system stiffness matrix
$[M]$	=	system mass matrix
N	=	axial or in-plane compressive periodic load
N_{cr}	=	buckling load
N_s	=	constant part of N
N_t	=	periodic part of N
α	=	N_s/N_{cr}
β	=	N_t/N_{cr}
$\{\delta\}$	=	as given in Eq. (7)
$\{\delta_1\}$	=	eigenvector of the dynamic stability problem
$\{\delta_2\}$	=	eigenvector of the free vibration problem
$\{\delta_3\}$	=	eigenvector of the buckling problem
θ	=	radian frequency of N_t
μ	=	as defined in Eq. (12)
Ω	=	θ/ω
ω	=	natural radian frequency

Introduction

PREDICTION of the dynamic stability behavior of structural members subjected to periodic loads is necessary for assessing the structural integrity of the structural members used in many fields of engineering. This study is absolutely necessary in the case of the rocket and missile structures, as a whole or its constituent structural members. The periodicity of the loads are not distinctly seen for the dynamic stability of these structures, as an ideal case of the simple combination of the applied constant compressive load and the periodic load. These ideal loads can be obtained by the harmonic (Fourier) analysis of the thrust time curve using a half-range cosine series. Thus a time-dependent thrust can be split into a constant compressive load and a series of periodic loads with decreasing periods as given by the harmonic analysis. The dynamic stability of the structural systems has to be analyzed at least for the combination of the constant part of the load and the periodic load with the highest period as this is the most critical periodic load for all practical purposes.

Earlier studies on the dynamic stability studies are briefly discussed by Timoshenko and Gere [1]. The theory and application of the dynamic stability behavior of structures have been exhaustively given in the classic work of Bolotin [2]. Although analytical methods to obtain the stability boundaries are attractive, for most of the structural members it is easier to obtain these boundaries by employing the approximate continuum methods or the powerful numerical method such as the finite element (FE) method. In either case, the governing equation of equilibrium is obtained in the form of a generalized matrix equation, which is the same for all structural members; the concerned matrices involved are the system stiffness, geometric, and mass matrices, the order of which varies depending on the structural member considered.

The first FE study on this topic for columns with various boundary conditions is by Brown et al. [3]. It is observed in this interesting study that when the first mode shapes of the free vibration and buckling are the same or nearly the same, which is generally satisfied for the many column boundary conditions, the dynamic stability regions more or less remain the same by proper nondimensionalization of the basic physical quantities in the problem involved. In this study [3], the authors chose these nondimensional parameters intuitively and the similarity of the free vibration and buckling mode shapes is an observation. As such, an assertive statement cannot be made about the generality of the insensitivity of the dynamic instability regions of the other structural members. For any other nondimensionalization of these basic quantities, a series of dynamic stability curves will be obtained and are relatively difficult for proper understanding of the dynamic stability behavior. A few studies on the dynamic stability for some column boundary conditions based on [3] can be seen [4,5] with the same nondimensionalization and obtain the master dynamic instability regions in either the analog or digital forms. The numerical results presented in these studies [3–5] show the near invariance of the dynamic instability boundaries. Some recent studies on this topic for plates and shells can be seen with different nondimensionalizations [6,7] and hence the problem of analyzing a number of dynamic instability curves as follows.

1) To prove rigorously the similarity of the mode shapes required in arriving at the intuitive nondimensionalization [3] followed in [4,5],

2) to derive a simple and accurate formula (master dynamic stability formula) for predicting the dynamic stability behavior of most of the structural members where the mode shapes of the fundamental frequency and the buckling load are the same or nearly the same, subjected to an end or edge periodic axial or in-plane loads, respectively,

3) to show the effectiveness of the master dynamic stability formula with reference to the column, plate, and shell problems, and

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4) to establish the invariance of the stability regions, using the nondimensional parameters derived here in the clear mathematical sense.

Formulation

When the structural members are subjected to an axial or in-plane loads of the form

$$N = N_s + N_t \cos\theta t \tag{1}$$

the stability boundaries can be obtained [2] following any approximate continuum or numerical methods from the solution of the following matrix equation of equilibrium, as

$$[K]\{\delta_1\} - \left(N_s \pm \frac{N_t}{2}\right)[G]\{\delta_1\} - \frac{\theta^2}{4}[M]\{\delta_1\} = 0 \tag{2}$$

Many researchers presented their new contributions on the dynamic stability of different structural members starting from this standard matrix equation. As the derivation of this equation is well documented in the classic work of Bolotin [2], and the objective of the present work is to derive a master dynamic stability formula, the authors also presented their new contribution starting from this matrix equation. However, interested readers may go to [2] for a better understanding of the derivation of Eq. (2).

From Eq. (2) the governing equation for the free vibration problem can be written, by neglecting the second term, as

$$[K]\{\delta_2\} - \omega^2[M]\{\delta_2\} = 0 \tag{3}$$

Equation (3) can be written as

$$[M]\{\delta_2\} = \frac{1}{\omega^2}[K]\{\delta_2\} \tag{4}$$

Similarly, the buckling problem can be solved from the degenerate case of Eq. (2) by neglecting the third term, as

$$[K]\{\delta_3\} - N_{cr}[G]\{\delta_3\} = 0 \tag{5}$$

$$[G]\{\delta_3\} = \frac{1}{N_{cr}}[K]\{\delta_3\} \tag{6}$$

or

Here, a reasonable assumption for the first node which is valid for most of the structural members can be made as

$$\{\delta_1\} = \{\delta_2\} = \{\delta_3\} = \{\delta\} \tag{7}$$

Further, it is emphasized here that a single term admissible function for the lateral displacement is used, in the case of approximate continuum methods, as a general practice for simplicity of the formulation, Eq. (7), is exactly satisfied. Otherwise this equation, in general, may be exactly or with minor difference satisfied, depending on the boundary conditions within the tolerable limits of engineering practice, and the error involved in the stability boundaries is negligible as has already been demonstrated in the study of the free vibrations of initially axial or in-plane loaded structural members [8,9].

Substituting Eqs. (4) and (6), using Eq. (7), in Eq. (2), we get

$$[K]\{\delta\} - \left(N_s \pm \frac{N_t}{2}\right)\frac{1}{N_{cr}}[K]\{\delta\} - \frac{\theta^2}{4}\frac{1}{\omega^2}[K]\{\delta\} = 0 \tag{8}$$

or

$$[K]\{\delta\} - (\alpha \pm \beta/2)[K]\{\delta\} - \frac{\theta^2}{4\omega^2}[K]\{\delta\} = 0 \tag{9}$$

where

$$\alpha = N_s/N_{cr} \quad \text{and} \quad \beta = N_t/N_{cr} \tag{10}$$

Equation (9) implies that

$$\frac{\theta^2}{4\omega^2} = 1 - (\alpha \pm \beta/2) = (1-\alpha)(1 \pm \mu) \tag{11}$$

where

$$\mu = \frac{\beta}{2(1-\alpha)} \tag{12}$$

Hence, the ratios of

$$\frac{\theta}{\omega} = \Omega = 2\sqrt{(1-\alpha)(1 \pm \mu)} \tag{13}$$

where Ω and μ are the nondimensional parameters used in [3] by intuition and are systematically obtained here. It is seen from Eq. (13) that these nondimensional parameters are independent of the characteristic values of the ω and N_{cr} of the structural members considered and can be used as a master dynamic stability formula as it is the most general form and is invariant for the structural members considered. However, to obtain the absolute values of the instability boundaries of a specific structural member, the corresponding values of ω and N_{cr} have to be used.

Numerical Results and Discussion

The formula derived in a rigorous way in the present Note to predict the stability boundaries of structural members subject to periodic loading is very general. In the nondimensional form the characteristic values of the structural members such as the fundamental frequency and the static buckling load parameters do not appear explicitly and hence are valid for any structural member even with complicating effects like shear deformation, rotary inertia, complex boundary constraints, effect of taper, material types used, etc. However, these characteristic values appear implicitly in the definitions of μ and Ω. The absolute stability boundaries for a specific structural member can be obtained by using the corresponding characteristic values available in handbooks. The advantage of using this formula is demonstrated with reference to the following structural members: a uniform column with any boundary condition (Fig. 1) subjected to an end periodic concentrated load, a layered square plate (Fig. 2) studied in [6] subjected to uniform edge periodic load, and a layered cylindrical shell (Fig. 3) with delamination subjected to uniform end periodic load [7].

For the column problem, the instability boundaries Ω_1 and Ω_2, between which it is dynamically unstable, are given with varying μ and for $\alpha = 0.0$, 0.5, and 0.8 in Table 1. Note that these values are independent of boundary conditions. The values of [3], obtained from the finite element analysis evaluated for any boundary conditions and reduced into the nondimensional forms μ and Ω for the same α values, digitized from the master stability curves, as named by the authors of [3], are also included in this Table. The digital form of presenting the results is chosen here to clearly demonstrate the numerical difference of the results between the

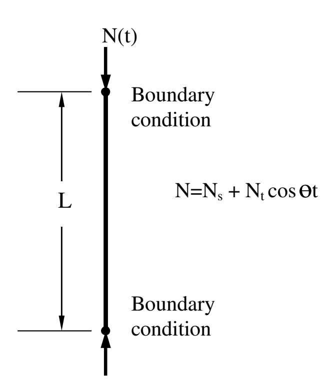

Fig. 1 A column subjected to a periodic end load.

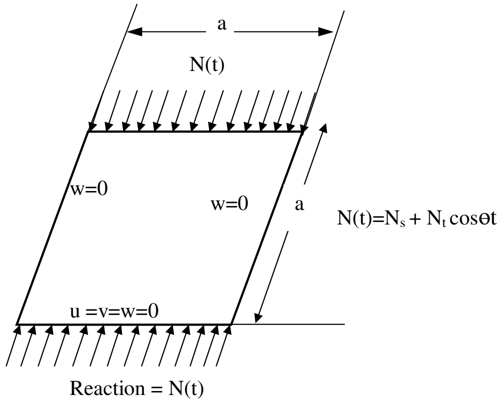

Fig. 2 Simply supported layered plate under a uniform uniaxial periodic compressive load.

present study and those of [3], as an analog form of presentation of the results will not show the small differences of the results of the two formulations. The authors tend to call the digital form of the results as the master dynamic stability tables. A close agreement between the present results obtained from the simple formula for the dynamic stability of the columns shows the effectiveness and universal application of the present simple formula.

Another problem considered is the dynamic stability of a layered square plate studied exhaustively by Dey and Singha [6] using the finite element method. The formulation takes care of the shear deformation and rotary inertia. The dynamic stability regions of the layered square plate are given in this study in terms of the nondimensional parameters [6] and the present results are deduced from the master dynamic stability table for $\alpha = 0$ as given in Table 2. The excellent agreement between the two results strongly indicates the usefulness of the present work.

The last problem considered is that of a layered cylindrical shell with delamination throughout the circumference (Fig. 3). The length and depth of the delamination and the end periodic load are shown in this figure. As the nondimensional forms used to obtain the dynamic stability curves [7] are different, the numerical results of the master stability table have been reduced to be consistent with this nondimensionalization. These results are shown in Table 3. Note the very good agreement between the present reduced results and those of [7].

These three problems considered show the usefulness and the general nature of the master stability table and are valid for the condition where the eigenvectors for the characteristic values of the structural members are the same or nearly the same. These three problems considered are of entirely different structural members and the successful application of the master dynamic stability formula and the subsequent master dynamic stability table simplifies the dynamic stability analysis by orders of magnitude.

Table 1 Variation of Ω_1 and Ω_2 for columns with end periodic load

	$\alpha = 0$		$\alpha = 0.5$		$\alpha = 0.8$	
μ	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
0	2.0000	2.0000	1.4142	1.4142	0.8944	0.8944
	(2.00)[a]	(2.00)	(1.40)	(1.40)	(0.91)	(0.91)
0.1	1.8973	2.0976	1.3416	1.4832	0.8485	0.9380
	(1.89)	(2.09)	(1.35)	(1.48)	(0.85)	(0.95)
0.2	1.7888	2.1908	1.2649	1.5491	0.8000	0.9797
	(1.79)	(2.18)	(1.27)	(1.56)	(0.79)	(0.98)
0.3	1.6733	2.2803	1.1832	1.6124	0.7483	1.0198
	(1.68)	(2.28)	(1.19)	(1.62)	(0.74)	(1.04)
0.4	1.5491	2.3664	1.0954	1.6733	0.6928	1.0583
	(1.57)	(2.37)	(1.12)	(1.69)	(0.69)	(1.08)
0.5	1.4142	2.4494	1.0000	1.7320	0.6324	1.0954
	(1.44)	(2.74)	(1.02)	(1.78)	(0.65)	(1.11)

[a]Values given in the parentheses are read from the graph [3].

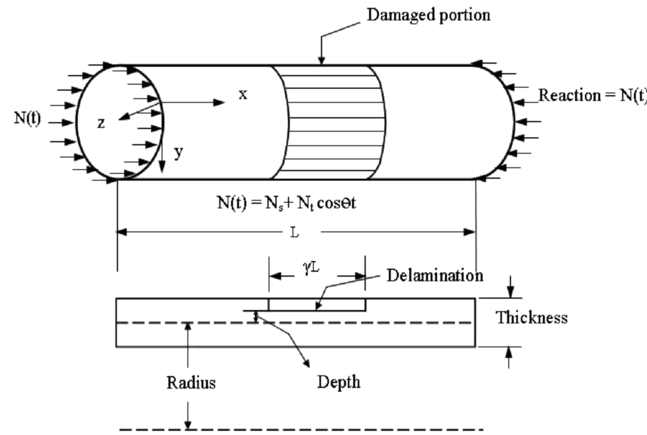

Fig. 3 Layered cylindrical shell with delamination under a uniform end periodic load.

Conclusions

A master dynamic stability formula is derived in the present study. In some of the earlier studies on this topic, the nondimensional parameters appearing in the formula are chosen by intuition. Whereas, in the present Note, the same nondimensional parameters are derived systematically in a rigorous mathematical sense. The assumption made in deriving this formula is that the mode shapes of the lowest characteristic values such as the frequency and buckling load parameters for most of the structural members are the same or nearly the same. This assumption is generally valid for the first mode of free vibration and buckling, which is of practical importance of many widely used structural members. However, other researchers who used the nondimensional parameters defined in the present work observed that the first free vibration and buckling mode are more or less the same. In this Note, it is shown that this condition is necessary to be used to derive the master dynamic stability formula which contains the nondimensional parameters identified by others intuitively. The effectiveness of the master dynamic stability formula and the master dynamic stability tables is demonstrated with reference to the dynamic stability of a column, a layered square plate,

Table 2 Variation of Ω_1 and Ω_2 for a layered square plate under uniform uniaxial edge periodic load for $\alpha = 0$

	Present study[a]		Dey & Singha[b] [6]	
β	Ω_1	Ω_2	Ω_1	Ω_2
0.0	2.0000	2.0000	2.00	2.00
0.25	1.8708	2.1213	1.87	2.13
0.5	1.7320	2.2360	1.74	2.24
0.75	1.5811	2.3452	1.57	2.34

[a]Deduced from the master dynamic stability table (present work)
[b]Values read from the graph.

Table 3 Variation of Ω_1 and Ω_2 of a layered cylindrical shell subjected to end uniform periodic load with circumferential delamination for $\alpha = 0.2$

	Present study[a]		Ref. [9][b]	
β	Ω_1	Ω_2	Ω_1	Ω_2
0.0	1.7888	1.7888	1.80	1.80
0.1	1.7320	1.8439	1.73	1.84
0.2	1.6733	1.8973	1.68	1.91
0.3	1.6124	1.9493	1.61	1.95
0.4	1.5491	2.0000	1.55	2.00
0.5	1.4832	2.0433	1.47	2.05

[a]Deduced from the master dynamic stability table (present work).
[b]Values read from the graph.

and a layered circular shell with delaminations, subjected to periodic loads. However, care should be exercised to use this formula indiscriminately without checking the similarity of the mode shapes. Further work is necessary to modify the present master dynamic stability formula to demonstrate its applicability if the mode shapes of the characteristic values are different. Finally, it is to be noted that the absolute values of the instability regions are dependant of the values of the frequency and buckling load of the structural members. Evaluation of these quantities is a standard procedure and can be easily obtained by supplying continuum or numerical methods or from the handbooks. The present formula derived can be effectively used with the complicating effects like taper, elastic foundation shear deformation, etc.

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